STABILITY OF A GYROSCOPE HAVING A VERTICAL AXIS OF THE OUTER RING WITH DRY FRICTION IN THE GIMBAL AXES TAKEN INTO ACCOUNT

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PMM Vol.24, No.3, 1960, pp. 568-571

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(Received 28 December 1959)

Rumiantsev in [1] investigates the influence of viscous friction on the stability of vertical rotation of a gyroscope on gimbals. In [2,3] the respective authors investigate the vertical rotation of a gyroscope taking into account dry friction in the suspension. The latter problem is further investigated in this paper.

Consider a gyroscope on gimbals as in Fig. 1.

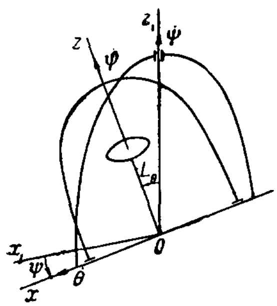

FIG. 1.

Let x_1, y_1, z_1 be the fixed coordinate system and xyz be the moving coordinate system attached to the casing (inner ring). Let the x-axis be along the axis of the casing, the z-axis along the spin axis. Let ψ be the rotation angle of the outer ring, θ the rotation angle of the casing in the outer ring, ϕ the spin angle of the rotor; let L denote the center of gravity of the casing together with the rotor, P their weight, A, B and C the moments of inertia of the gyroscope about the moving axes x, y and z, respectively. A_1, B_1 and C_1 the moments of inertia of the casing about the same moving axes x, y and z, respectively, J the moment of inertia of the outer ring about the z_1-axis. Let the z_1-axis coinciding with the axis of the outer ring be vertical, and let the distance OL equal ζ.

Along the x-axis acts the moment of dry friction $M_1 = - B_1 \operatorname{sign} \theta$, where $B_1 > 0$, and along the z-axis acts the moment $M_2 = - B_2 \operatorname{sign} \psi$, where $B_2 > 0$.

Since the variables ψ, θ and ϕ are independent holonomic variables, we can write the equations of motion of the system in the form of Lagrange equations of the second kind

$$
\begin{gathered}
(A+A_1)\theta^{\cdot\cdot}-(A+B_1-C_1)\psi^{\cdot 2}\sin\theta\cos\theta+C(\varphi^{\cdot}+\psi^{\cdot}\cos\theta)\psi^{\cdot}\sin\theta-P\zeta\sin\theta=M_1 \\
\frac{d}{dt}[J\psi^{\cdot}+(A+B_1)\psi^{\cdot}\sin^2\theta+C_1\psi^{\cdot}\cos^2\theta+C(\varphi^{\cdot}+\psi^{\cdot}\cos\theta)\cos\theta]=M_2 \qquad (1) \\
\frac{d}{dt}(\varphi^{\cdot}+\psi^{\cdot}\cos\theta)=0
\end{gathered}
$$

The last equation, corresponding to the cyclic coordinate ϕ, yields the first integral

$$\varphi^{\cdot}+\psi^{\cdot}\cos\theta=r_0$$

Let us introduce a unit sphere whose center coincides with the fixed point O and investigate trajectories of the point of intersection of the axis of the rotor with the surface of the sphere.

According to Painlevé [4], the value of M_1 when $\dot{\theta}=0$ lies in the interval $[-B_1, +B_1]$, and the value of M_2 when $\psi=0$ lies in the interval $[-B_2, +B_2]$. Further

$$
M_1|_{\theta^{\cdot}=0}=\begin{cases}-B_1 & \text{for } f_1(\psi^{\cdot},\theta)<-B_1\\ B_1 & \text{for } f_1(\psi^{\cdot},\theta)>B_1\\ f_1(\psi^{\cdot},\theta) & \text{for } |f_1(\psi^{\cdot},\theta)|\leqslant B_1\end{cases} \qquad (2)
$$

$$
M_2|_{\psi^{\cdot}=0}=\begin{cases}-B_2 & \text{for } f_2(\theta^{\cdot},\theta)<-B_2\\ B_2 & \text{for } f_2(\theta^{\cdot},\theta)>B_2\\ f_2(\theta^{\cdot},\theta) & \text{for } |f_2(\theta^{\cdot},\theta)|\leqslant B_2\end{cases} \qquad (3)
$$

where

$$
\begin{gathered}
f_1(\psi^{\cdot},\theta)=-(A+B_1-C_1)\psi^{\cdot 2}\sin\theta\cos\theta+Cr_0\psi^{\cdot}\sin\theta-P\zeta\sin\theta \\
f_2(\theta^{\cdot},\theta)=-Cr_0\theta^{\cdot}\sin\theta \qquad (4)
\end{gathered}
$$

Each of the first two inequalities (2) corresponds, according to (1), to the presence on the trajectory of one point $\dot{\theta}=0$ for which $\ddot{\theta}\neq 0$. The third inequality (2) gives $\ddot{\theta}=0$, which corresponds to the part of the trajectory lying on a parallel of latitude. From the system (1) follows that the first two inequalities (3) imply that $\ddot{\psi}\neq 0$ and the third inequality (3) leads to $\ddot{\psi}=0$, which corresponds to the part of the trajectory lying on a meridian.

Let us take the segment of the trajectory which lies on the parallel of latitude

$\theta=\theta_1$, $\dot{\theta}=0$, $\dot{\psi}_1\neq 0$ (ψ_1, θ_1 are at the initial point of the segment)

which is represented by the equations

$$(J+(A+B_1)\sin^2\theta_1+C_1\cos^2\theta_1)\psi^{\cdot\cdot}=\pm B_2, \qquad |f_1(\psi^{\cdot},\theta_1)|\leqslant B_1 \tag{5}$$

The last inequality gives $\dot{\psi}^{\circ} < \dot{\psi} < \dot{\psi}^{\circ\circ}$, where $\dot{\psi}^{\circ}$, $\dot{\psi}^{\circ\circ}$ are roots of equation $|f_1(\psi, \theta_1)| = B_1$ (See Fig. 2). The angular velocity is a linear function of the time t

$$\psi^{\cdot}(t)=\begin{cases}\psi_1^{\cdot}-\dfrac{B_2 t}{J+(A+B_1)\sin^2\theta_1+C_1\cos^2\theta_1} & \text{при } \psi_1^{\cdot}>0\\ \psi_1^{\cdot}+\dfrac{B_2 t}{J+(A+B_1)\sin^2\theta_1+C_1\cos^2\theta_1} & \text{при } \psi_1^{\cdot}<0^{\cdot}\end{cases}$$

Let us assume for the sake of simplicity that the initial instant of time $t = 0$ occurs at the initial point of the segment (ψ_1, θ_1), that is $\dot{\psi} = \dot{\psi}_1$, at $t = 0$. It can be easily shown that the trajectory could lie on a parallel of latitude only in a final interval of time $[0, T_1]$ where

$$T_1=\frac{J+(A+B_1)\sin^2\theta_1+C_1\cos^2\theta_1}{B_2}|\psi_1^{\cdot}|$$

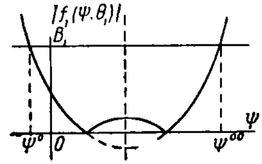

FIG. 2.

When $t = T_1$ the gyroscope falls into a "stationary zone". Beginning from this instant of time, the axis of the rotor occupies a certain fixed position in space (ψ = constant, θ = constant). It means that in such a case the regular precession, which is possible when the forces are conservative, is absent. Thus, when dry friction is taken into account, the class of characteristic motions becomes smaller. We shall now investigate the trajectory segment along the meridian

$$\psi = \psi_1, \quad \dot{\psi} = 0, \quad \ddot{\psi} = 0, \quad \dot{\theta}_1 \neq 0$$

(ψ_1, θ_1 are at the initial point of the segment)

which is represented by the equation

$$(A+A_1)\theta^{\cdot\cdot}-P\zeta\sin\theta=\pm B_1, \qquad |f_2(\theta^{\cdot},\theta)|\leqslant B_2 \tag{6}$$

Let us designate

$$\frac{P\zeta}{A+A_1}=a, \qquad \frac{B_1}{A+A_1}=b_1$$

We shall demonstrate now that after a finite interval of time T_2 the gyroscope will fall into a "stationary zone", meaning that the velocity $\dot{\theta}$ vanishes when $t = T_2$. Let $\theta_1 > 0$. We shall replace in (6)

$$\theta^{\cdot}=f(\theta), \qquad \theta^{\cdot\cdot}=\frac{df}{d\theta}f$$

We obtain then

$$\frac{1}{2}(f^2 - f_1^2) = a(\cos\theta_1 - \cos\theta) + b_1(\theta_1 - \theta), \qquad f_1 = f(\theta_1) = \theta_1^{\cdot}$$

Let $\theta_T^{\cdot}$ be a root of the equation

$$\theta_1^{\cdot 2} + 2a(\cos\theta_1 - \cos\theta) + 2b_1(\theta_1 - \theta) = 0$$

Then the desired value of T_2 could be expressed in the form

$$T_2 = \left| \int\limits_{\theta_1}^{\theta_T} \frac{d\theta}{\sqrt{K - 2a\cos\theta - 2b_1\theta}} \right|, \qquad K = \theta_1^{\cdot 2} + 2a\cos\theta_1 + 2b_1\theta_1$$

We shall investigate now the vertical rotation

$$\theta = 0, \qquad \theta^{\cdot} = 0, \qquad \psi^{\cdot} = 0, \qquad r_0 = \omega \tag{7}$$

The system (1) shows that such a rotation is possible, and on the strength of (2), (3) and (4)

$$M_1 = 0 \quad \text{при } \theta^{\cdot} = 0, \quad \theta = 0, \qquad M_2 = 0 \quad \text{при } \psi^{\cdot} = 0, \quad \theta = 0$$

We shall study the stability of this motion. For this purpose we shall investigate as usual the perturbed motion

$$\theta = \eta_1, \qquad \theta^{\cdot} = \xi_1 = \eta_1^{\cdot}, \qquad \psi^{\cdot} = \xi_2, \qquad r_0 = \omega + \xi_3$$

On the strength of the above considerations, we shall study separately the behavior of the perturbed trajectories lying on the three segments

1. $\xi_1 \neq 0, \ \xi_2 \neq 0$, **2.** $\xi_1 = 0, \ \xi_2 \neq 0$, **3.** $\xi_1 \neq 0, \ \xi_2 = 0$

For each segment, we could construct a sign-definite function in the form of a linear combination of integrals, obtained from the corresponding equations of the perturbed motion.

1. On the first segment we have a relation similar to the law of conservation of energy when the forces are conservative

$$J\psi^{\cdot 2} + (A + A_1)\theta^{\cdot 2} + (A + B_1)\psi^{\cdot 2}\sin^2\theta + C_1\psi^{\cdot 2}\cos^2\theta + 2P\zeta\cos\theta + \\ + Cr_0^2 - 2M_1\theta - 2M_2\psi = h_1$$

The equation of the perturbed motion yields two integrals

$$W_1 = \xi_3, \quad W_2 = (A + A_1)\xi_1^2 + (J + C_1)\xi_2^2 + C(\xi_3^2 + 2\omega\xi_3) - P\zeta\eta_1^2 - 2M_1\eta_1 - 2M_2\psi \tag{8}$$

The angle denoted previously by ψ is marked here by the letter η_2 in order to indicate that we do not investigate stability with respect to ψ.

We shall introduce the function

$$\Phi_1 = 2\,(M_1\eta_1 + M_2\psi)$$

and construct the combination

$$V_1 = \Phi_1 + W_2 - 2C\omega W_1$$

When $\zeta < 0$ the function V_1 is a positive-definite quadratic form. The time derivative of V_1 equals, on the strength of the equations for the perturbed motion

$$\frac{dV_1}{dt} = \frac{d\Phi_1}{dt} = 2\,(M_1\xi_1 + M_2\xi_2)$$

and since

$$M_1\xi_1 = -B_1\xi_1 \operatorname{sign}\xi_1 < 0, \qquad M_2\xi_2 = -B_2\xi_2 \operatorname{sign}\xi_2 < 0$$

it is negative.

2. The equations of the perturbed motion for the second segment are given by (5), that is

$$(J + (A + B_1)\sin^2\theta_1 + C_1\cos^2\theta_1)\frac{d\xi_2}{dt} = \pm B_2, \qquad |f_1(\xi_2, \theta_1)| \leqslant B_1 \tag{9}$$

In view of the fact that the friction M_1 on the considered segment $\theta = \theta_1$ does not do any work, the equations of the perturbed motion admit the integral

$$W_2' = (J + (A + B_1)\sin^2\theta_1 + C_1\cos^2\theta_1)\,\xi_2^2 + C\,(\xi_3^2 + 2\omega\xi_3) - 2M_2\psi$$

The integral $W_1 = \xi_3$ remains unchanged. We shall construct the combination

$$V_2 = \Phi_2 + W_2' - 2C\omega W_1, \qquad \Phi_2 = 2M_2\psi$$

The time derivative of V_2 is, on the strength of (9)

$$\frac{dV_2}{dt} = \frac{d\Phi_2}{dt} = 2M_2\xi_2 < 0$$

3. The equations of the perturbed motion on the third segment are

$$(A + A_1)\frac{d\xi_1}{dt} - P\zeta\eta_1 = \pm B_1, \qquad \frac{d\eta_1}{dt} = \xi_1, \qquad |f_2(\xi_1, \eta_1)| \leqslant B_2 \tag{10}$$

and admit the integral

$$W_2'' = (A + A_1)\,\xi_1^2 + C\,(\xi_3^2 + 2\omega\xi_3) - P\zeta\eta_1^2 - 2M_1\eta_1$$

Under the condition $\zeta < 0$ the combination

$$V_3 = \Phi_3 + W_2'' - 2C\omega W_1, \qquad \Phi_3 = 2M_1\eta_1$$

is a positive-definite function, whose time derivative is, on the strength of (10), negative

$$\frac{dV_3}{dt} = \frac{d\Phi_3}{dt} = 2M_1\xi_1 < 0$$

In this way, to every segment of the perturbed trajectory there corresponds a function which becomes positive-definite when the following condition is satisfied:

$$\zeta < 0 \tag{11}$$

The time derivative of every function is negative on the strength of the corresponding system of equations for the perturbed motion. The investigated motion (7) is stable under the above conditions, and we have asymptotic stability with respect to ξ_1, and ξ_2. This requires additional explanation.

We must show that for any chosen number $\epsilon > 0$ we can find such a number $\delta > 0$, that if the initial perturbations are inside the sphere δ

$$\eta_1^{\circ 2} + \xi_1^{\circ 2} + \xi_2^{\circ 2} + \xi_3^{\circ 2} \leqslant \delta$$

then the motion of the perturbed trajectory will always be inside the sphere ϵ.

$$\eta_1^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 \leqslant \varepsilon$$

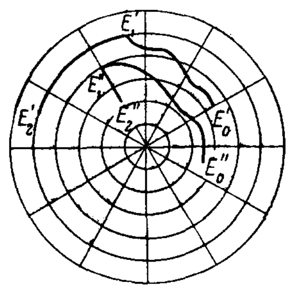

FIG. 3.

From the above considerations follows that the perturbed trajectories caused by small initial perturbations can be of two types (Fig. 3). The trajectories are projected on the plane tangent at the North pole to the unit sphere with its center at 0. The initial points are E_0' and E_0''. The velocity ξ_1 vanishes at the point E_1'. ξ_2 vanishes at E_2' and the gyroscope falls into the "stationary zone".

The trajectory segment $E_1' E_2'$ coincides with a parallel of latitude. At the point $E_1'' \xi_2$ vanishes at the point $E_2'' \xi_1$ vanishes and the gyroscope falls again into the "stationary zone". The trajectory segment E_1'' E_2'' coincides with a meridian.

For a given $\epsilon > 0$ we could construct $\delta_1 > 0$, $\delta_2 > 0$ corresponding to the two types of trajectories. Then, the desired value of δ is

$$\delta = \min(\delta_1, \delta_2)$$

Using the functions V_1, V_2, V_3, we make an estimate of δ and obtain

$$\delta = \min\left\{\frac{K_2K_2'}{K_1(K_1' + \varkappa)}\,\varepsilon, \quad \frac{K_2K_2''}{K_1K_1''}\,\varepsilon\right\}$$

where

$$K_1 = \max(A + A_1,\ J + C_1,\ C,\ -P\zeta), \qquad K_2 = \min(A + A_1,\ J + C_1,\ C,\ -P\zeta)$$

$$K_1' = \max(J + C_1,\ C) \qquad K_2' = \min(J + C_1,\ C)$$

$$K_2' = \max(A + A_1,\ C,\ -P\zeta), \qquad K_2'' = \min(A + A_1,\ C,\ -P\zeta), \quad \varkappa = \frac{K_2'(K_2' - K_2)}{K_2}$$

It means that the investigated motion is stable.

The result which we obtained could be generalized. Let us consider the motion

$$\theta = \theta_0 \qquad \theta^{\cdot} = 0, \qquad \psi^{\cdot} = 0, \qquad r_0 = \omega \tag{12}$$

which represents constant rotation about an axis at an angle $\theta = \theta_0$ to the vertical. With friction absent ($M_1 \equiv 0$, $M_2 \equiv 0$) such a motion is impossible according to (1). The conditions (2) and (4) show that under the condition

$$|P\zeta \sin\theta_0| \leqslant B_1 \tag{13}$$

such a motion is possible. Therefore, the inequality (13) represents the condition for the existence of the motion (12).

Investigation of stability of this motion could be carried through similarly. The sufficient condition for stability could be reduced to the inequality (11).

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Translated by T.L.